Errata List for “A Taste of Jordan Algebras”

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Correction to the title of the book!! Ivan Shestakov has belatedly revealed that Pascual Jordan came from an old Spanish family who emigrated to Germany during the Napoleonic wars. The original family name was Jorda (pronounced “Hchh-orda”, as the Spanish J is aspirated like ch in the Scottish “loch” or the German “ich”), and the first-born son was traditionally named Pascual. The family name was eventually Germanicized to a proer Jordan (“Yorr-dahn”), but by rights the algebras should be known as \textit{Jorda algebras}. I think it is too late to repair the damage, and by abuse of language we will call them \textit{Jordan algebras}.

When I first showed my hot-off-the-press textbook to my mother, decidedly a non-mathematician, I condescendingly watched her read the first page of the book, expecting her to get lost after a few paragraphs. But within a minute she said “Look, a typo on page 1 of the preface!” It’s been all downhill from there.

I want to thank the University of Ottawa Algebra Seminar 2004-5 [organized by Erhard Neher and Maribel Tocon, with Samina Bashir, Sergei Krutelevich, Michael Lau, and Angelika Welte] for a very careful reading of much of the text, and (for which I’m not so thankful) coming up with an embarrassingly long list of errata. More recently, my Spring 2006 Jordan Algebra class this has noticed a few additional errors. In 2008 Svante Janson found several mathematical corrections as well as a medical one. What follows is the complete list of errata which have been brought to my attention, from whatever source, as of January 2008. Items are listed by page p. and line L.; the emended portion is indicated in boldface type, and any explanatory or exculpatory comment is in italics.

I have broken the errata into three groups, depending on the erratic degree (somewhat akin to the system for measuring degree of severity of burns: first degree is a bad sunburn, but third-degree is life-threatening). First degree errata are minor (I bet you never even noticed them), often linguistic typos, and you can safely let them lie slightly below the level of consciousness. Second degree errata are mathematically wrong, but you probably quickly and easily made the correction yourself. Third-degree errata are mathematically wrong and require some surgery to correct; a patch must be inserted, and I give some descriptive commentary as I make the incision.

Third-Degree Errata

p. 134 L.-7 of characteristic \( p \) when \( \delta^n = 0 \) for \( n < (p + 2)/2 \).

\textit{I believed tenaciously ever since graduate school that when }\delta^n = 0 \textit{ the exponential }\exp(\delta) \textit{ is guaranteed to be an automorphism if } n! \textit{ is invertible, but in fact for }\exp(t) \textit{ to be an automorphism of }A[t] \textit{ for generic } t \textit{ it is necessary and sufficient that } (2(n - 1))! \textit{ be invertible (so this is always sufficient for }\exp(\delta) \textit{ alone to be an automorphism of } A). \textit{ In characteristic } p \textit{ this means } 2(n - 1) < p, n < \frac{p + 2}{2}.

\textit{Thus }Exercise 1.2.2(3) \textit{ becomes a nontrivial digressive exercise in itself: EXERCISE 1.2.2B(i) Let }A \textit{ be a linear algebra over } \Phi. \textit{Convince yourself that if } A \textit{ has no } \alpha \textit{-torsion for } \alpha \in \Phi \textit{ (}ax = 0 \implies x = 0, \textit{i.e., } \alpha 1_A \textit{ is injective), then } A \textit{ imbeds in a scalar extension } A \textit{ with } \alpha \textit{ invertible in } (\textit{so } \alpha 1_A \textit{ is bijective), and that any derivation or automorphism of } A \textit{ extends uniquely to one on } A. \textit{ This allows us pass from absence-of-torsion to presence-of-inverse. (ii)
Note that $n!$ is invertible if $p$ is invertible for all primes $p = n$ (so $2!$ requires $2$; $3!$ and $4!$ require $2, 3, 5!$ and $6!$ require $2, 3, 5; 6$; etc.), and that in prime characteristic $p n!$ is invertible for all $n < p$.

We will say that $n$ is an invertible index if $n!$ is invertible. (iii) If $\delta$ is a derivation of a linear algebra $A$, nilpotent of index $n, \delta^n = 0$, and if $n - 1$ is an invertible index, then the exponential series $E := \exp(\delta) := \sum_{k=0}^{\infty} \frac{\delta^k}{k!} = \sum_{k=0}^{\infty} \frac{\delta^k}{k!}$ makes sense. We define the divided power $\delta_k := \delta^k/k!$ if $k - n - 1$, and $\delta_k := 0$ if $k \geq n$, so $E$ becomes simply $\sum_{k=0}^{\infty} \delta_k$ without any annoying coefficients. Show that $E$ is an invertible linear transformation, and $\delta_i \delta_j = (\ell^{ij}) \delta_{i+j}$ for all $i, j \geq 0$. (iv) Show that the $E(\alpha \beta) := \exp(\alpha \beta) := \sum_{k=0}^{\infty} \frac{\alpha \beta \delta_k}{k!}$ satisfy $E(\alpha + \beta) = E(\alpha)E(\beta)$, so we have a 1-parameter group of bijections of $A$. (v) The question of when $E(t)$ for generic $t$ is an automorphism of $A[t]$ is more delicate. Show that for an arbitrary derivation, with no nilpotence and no condition on scalars, for all $k$ and all $x, y \in A$ we have $\delta^k(xy) = \sum_{i=0}^{k} (\ell^{ij}) \delta^i(x) \delta^{k-i}(y)$, so that if $k$ is an invertible index then $\delta_k(xy) = \sum_{i,j \geq 0, i+j=k} \delta_i(x) \delta_j(y)$. (vi) Show that if $\delta^n = 0$ for invertible index $n - 1$, then $E(t)$ is a derivation iff for all $x, y \in A$ each coefficient of $t^k$ vanishes in $E(t)(x)E(t)(y) - E(t)(xy) = \sum_{k=n}^{2n-2} t^k \sum_{i,j} \delta_i(x) \delta_j(y)$ for

\[
\Delta_k^u(x, y) := \sum_{i,j \leq n, i+j=k} \delta_i(x) \delta_j(y).
\]

Compute $\Delta_k^u(x, y)$ for $n = 2, k = 2$; $n = 3$, $k = 3, 4$; $n = 4, n = 4, 5, 6$; $n = 5, k = 5, 6, 7, 8$. (vii) In the situation of (vi), show that $k! \Delta_k^u(x, y) = 0$. Conclude that if $2n - 2$ is an invertible index then $E(t)$ is a derivation of $A[t]$ (hence $E$ is a derivation of $A$).

Then one would add to Appendix E p. 506:

**EXERCISE 1.2.2B:** (i) $E := 1_A - Z$ for a nilpotent $Z = \delta f(\delta)$. (iii) Trivial if $i + j \geq n$, by definition of binomial coefficients if $i+j < n$ so $(i+j)!$ exists. (iv) $\alpha \delta$ is again a nilpotent derivation of index $n$: $E(\alpha \beta) := \sum_{i \leq n} \binom{n}{i} \alpha^i \beta^{n-i}$, and $E(\alpha \beta) = \sum_{i \leq n} \alpha^i \beta^{n-i} \delta_i$. (v) Induct on $k$. (vi) $E(t)$ is an automorphism of $A[t]$ for each $x, y$ all the coefficients of $t^k$ in $E(t)(x)E(t)(y) - E(t)(xy)$ vanish. If $\delta^n = 0$ then we have that $E(t)(x)E(t)(y) = \sum_{k=0}^{n-1} t^k \delta_k(x) \sum_{j=n-1}^{1} t^j \delta_j(y)$

\[
= \sum_{k=0}^{n-1} t^k \sum_{i,j \leq n, i+j=k} \delta_i(x) \delta_j(y) = \sum_{k=0}^{n-1} t^k \Delta_k^u(x, y)
\]

and also that $E(t)(x) = \sum_{k=0}^{n-1} t^k \delta_k(x)$ since $E$ is an invertible index, then the exponential $E := \exp(\delta)$ is a derivation of a linear algebra $A$, nilpotent $\delta^n = 0$, and invertible $n - 1$.

Compute $\Delta_2 = \delta(x) \delta(y); \Delta_3 = \delta(x) \delta_2(y) + \delta_2(x) \delta(y); \Delta_4 = \delta(x) \delta_3(y) + \delta_2(x) \delta(y) + \delta_3(x) \delta(y); \Delta_5 = \delta_2(x) \delta_3(y) + \delta_3(x) \delta(y); \Delta_6 = \delta_2(x) \delta_4(y) + \delta_3(x) \delta(y); \Delta_7 = \delta_2(x) \delta_5(y) + \delta_3(x) \delta(y) + \delta_4(x) \delta(y); \Delta_8 = \delta_3(x) \delta_4(y) + \delta_4(x) \delta(y); \Delta_9 = \delta_3(x) \delta_5(y) + \delta_4(x) \delta(y) + \delta_5(x) \delta(y); \Delta_{10} = \delta_4(x) \delta_5(y)$. (vii) From (vi) the noninvertible indices $k \neq 2n - 2$ have $0 = 2t^k \delta^k(xy) = \sum_{i,j \leq n, i+j=k} \binom{n}{i,j} t^i \delta^i(x) \delta^{n-i}(y)$

\[
= \sum_{k=0}^{n-1} t^k \delta_k(x) \delta_k(y) = k! \Delta_k^u,
\]

and if $2n - 2$ is invertible is $k$ and we can cancel $k!$.

p. 160 L.8 

\[
(\xi_1 \times \xi_2)(\eta_1 \times \xi_2) = (\xi_1 \times \xi_2)(\eta_1 \times \xi_2)
\]

It is crucial that the middle products have $\times$ in place of $\cdot$.

p. 160 L.11 and p. 167 L.29 

[Use the Moufang identities 21.1 (1)-(3) p. 336.]

**EXERCISE 2.4.2B, Problem 2.1, 2.2** (indeed, almost all calculations in alternative algebras) require the Moufang identities. These are instantly loaded into memory whenever one encounters alternative algebras, a dangerous habit for writers of textbooks.

p. 180 L.4 

$+\sigma'(x, y) + \sigma'(y, c)x - \sigma'(x, y) c$

p. 180 L.7 

$\sigma'(x, y) = \alpha \beta - \sigma(u, v)$

Recall from the Remark on p. 179 that there is tension between the global $J\text{ord}(Q, c)$ construction, where in $x \bullet y$ the coefficient of $1$ is $-Q(x, y)$, and the $J\text{Spin}(M, \sigma)$ construction, where the coefficient is $+\sigma'(x, y)$. Well, in this Exercise that tension got the best of me: in (1), (2) I tried to have
\(Jord(\sigma, c)\) follow the \(J\hbox{Spin}(M, \sigma)\) model, with \(\sigma'\) a global extension of the original \(\sigma\) instead of an extension of \(-\sigma\), but in (4) I wanted the \(\sigma'\) construction to follow \(Jord(Q, c)\) with the coefficient of 1 being \(-\sigma'(x, y)\) and \(\sigma'(x, y) = \alpha\beta - \sigma(u, v)\) being a global extension of \(-\sigma\). I can't have it both ways.

p. 232 L.16 \((A_u, *) \rightarrow (A, *_u)\)

You can isotope the algebra but keep the involution, or keep the algebra but isotope the involution, both are equivalent. The new involution has hermitian elements \(uH = Hu^{-1}\) since \(uHu = H\) when \(u^* = u\). The inverse \(L^{-1}_u : *_u \rightarrow *\) is an isomorphism of involutions, but on the level of algebras it is an isomorphism \(A \rightarrow A_u\).

p. 243 L.12 \(\text{norm: show when } \Phi \text{ has no nilpotent elements that}\)

The conclusion from given \(T, Q\) holds for all scalars, but the conclusion about \(T, Q\) only holds in the absence of nilpotent scalars.

p. 350 L.-5,-4 This is just a variant of \(G_9\). This makes it crystal clear that \(G_8\) implies this variant of \(G_9\).

Computer calculation seems to indicate there are 6 non-equivalent \(s\)-identities of degree 9, and Shestakov's identity is definitely not Glennie's \(G_9\). This applies to page 470 as well.

p. 412 L.18 Whatever happened to a proof of Strong Modulus Exclusion (4)? This is a nontrivial result, depending on the deep Absorber Nilness Theorem 4.3.1. If \(c \in \mathcal{J}_1(qA(B))\) then by Absorber Nilness \(c^n = b \in B\) for some \(n\); in particular, if \(c\) is a modulus for \(B\) then so is \(b\) by (1), lying inside \(B\), so by (3) we must have \(B = J\).

p. 470 L.1 Both \(G_8\) and Shestakov's variant of \(G_9\).

p. 495 L. 5 associative algebra over a faithful ring of scalars as in C.4.1. On the top of pages 496 and 497 I use the scalar relation \(n(ab) = n(a)n(b)\), although C.4.2 p. 491 only establishes the elemental relation \((A10) n(ab)1 = n(a)n(b)1\). As in C.2.4 p. 485, an elemental condition \(\alpha1 = 0\) implies \(\alpha = 0\) if \(\varphi\) is faithful (e.g. if it has no 3-torsion or no nilpotents, as on the top of page 486). Note that in C.4.1-4 we can always replace a given \(\varphi\) by \(\varphi/\text{Ann}(A)\) and assume from the start that is faithful.
Second-Degree Errata

p. 11 L.-10,-11 subspace $\text{Der}(C)$ of derivations of $C$

p. 52 L.10 I should have reminded the reader that a bilinear product $A \times A \to A$ is the same as a linear map $A \otimes A \to A$ on the tensor product.

p. 140 L.-12 and scalar extension commutes with the exchange functor.

Clearly unitalization does not commute with $E x$: $\widehat{E x(A)}$ adds one dimension to $E x(A)$, while $E x(\widehat{A})$ adds two.

p. 151 L.11 homomorphism $\widehat{A} \to B$.

p. 156 L.3-13 The notation $Q(x,y)$ is not defined until line 14.

p. 162 L.14 is alternative iff $A$ is associative

The “if” part is alluded to at the end of the proof, to be established by the reader in the exercise.

p. 173 L.24 $zx = x^*z$ for all "Remove the minus sign.

p. 173 L.24 $(x^*hx)a$ Move the $*$ to the first $x$.

The abbreviations $\mathcal{H}$ and $\widehat{\mathcal{H}}$ could have been better explained: in line 22 state “for $\mathcal{H} := \mathcal{H}(A, \ast)$”, in line 23 say “$h \in \widehat{\mathcal{H}} := H + \Phi 1$”. 

p. 183 L.18 $1_J + \varepsilon D (\varepsilon$ a dual number)

p. 206 L.4 $B_{\alpha,x,y}U_zB_{\alpha,y,x}$ Switch the $x, y$ on the right side of the equation.

p. 288 L.20 $f(a[21], 1[12], c[13], 1[31], b[23]) = \text{Switch the order of the last 3 terms.}$

p. 336 L.3 $= (xy)xz$

p. 365 L.19 $z \mapsto U_zy$

p. 414 L.19 Recall that invertibility of $\widehat{1} - c$ guarantees $U_{1-c}J = J$ (not just $U_{1-c}\widehat{J} = \widehat{J}$) by Basic Quasi-Inverse Theorem II.1.3.2 (3iii).

p. 510 L.-3 $U_zU_aU_z\widehat{J}$
First-Degree Errata

p. i  L.-3  to capture the colloquial
p. 55  L.10  Thus a subdirect product
p. 112  L.5  *The third matrix should have 31 or 13 entry \(-j\) instead of \(j\).*

p. 118  L.5  \(\{c, B\} \subseteq B;\) it (delete close parenthesis)

p. 130  L.4  Part 1 A Historical
p. 144  L.-5  is not centroidal

p. 145  L.9  claim that it \{ will suffice \} \langle reader’s choice \rangle

p. 170  L.-11  \([\hat{\Lambda}^+] = (\hat{\Lambda}^+)\) \(\text{The} \) is easily overlooked.

p. 198  L.7  (3) Show
p. 205  L.-1  too little
p. 247  L.5  Example 8.3.3
p. 252  L.7  \(= 0\),

p. 413  L.-10  two-sided ideal contained in \(S\)

p. 527  L.-7  arbitrary fields as it does for
p. 532  L.2  [dikk brukk]

Professor Kegel wrote me plaintively asking why I called R.H. Bruck ralf when all his friends called him dikk. I hope no other friends of R. H. Bruck have noticed this gaffe, possibly resulting from a confusion between Ralph Fox and Richard D. Bruck, who both were at Wisconsin at some time in the distant past.

p. 532-33  The text went to print without one final TeX pass, and the page references O from the Introduction got left out (replaced by ??) in the citation of the Pronouncing Index. This applies to all the people I carefully thanked: John Faulkner, Florie and Nathan Jacobson, Wilhelm Kaup, and Kurt Meyberg. Their citations are gone, but not forgotten!

Stay tuned for the latest developments