Recent Advances
in
Modular Representation Theory
of
Hecke Algebras (III)

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§1 The canonical basis

Recall that $V_{\nu}(\Lambda)$ is an integrable highest weight module, $(L, B)$ is its crystal basis defined by Kashiwara.

**Definition 1.** The canonical (or global) basis $B(\Lambda)$ of $V_{\nu}(\Lambda)$ is the set of elements $\{G_{\nu}(b)\}_{b \in B}$ which are uniquely determined by

1. $G_{\nu}(b) \in L$.
2. $G_{\nu}(b) + \nu L = b$ in $L/\nu L$.
3. $\overline{G_{\nu}(b)} = G_{\nu}(b)$, that is, $G_{\nu}(b)$ is fixed by the semilinear involution of $V_{\nu}(\Lambda)$ defined by

   $\begin{cases}
   (a) & \overline{m_{\Lambda}} = m_{\Lambda}, \\
   (b) & \overline{f_i m} = f_i \overline{m}, \text{ for all } m \in V_{\nu}(\Lambda).
   \end{cases}$

Compare this with the definition of the Kazhdan-Lusztig basis. The basis is the canonical basis of the Hecke algebra $\mathbb{H}_W(q)$.  

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Take the Kostant-Lusztig $\mathbb{Z}[v, v^{-1}]$-lattice in $U_v^-$ and multiply the lattice with $m_\Lambda$.

Specializing them at $v = 1$, we get the integrable $\mathfrak{g}$-module $V(\Lambda)$ and its basis $\{G'(b)\}_{b \in B(\Lambda)}$, which we call the canonical basis again.

We also reduce the deformed Fock space $\mathcal{F}_v$ to

$$\mathcal{F} = \bigoplus_{\lambda \in \mathcal{P}^m} \mathbb{Q}\lambda \supset U(\mathfrak{g})\emptyset \simeq V(\Lambda).$$

It does not matter whether we consider the tensor product action or the JMMO action: the resulting $\mathfrak{g}$-module structure on $\mathcal{F}$ is the same. Hence the set $\{G(b)\}_{b \in B(\Lambda)}$ is also the same. What is different is the labelling of the basis elements $G(b)$. The labelling is important because we attach a Hecke algebra module to each of the canonical basis elements.
§2 The decomposition numbers

Below is my old result on the decomposition numbers of $\mathcal{H}_n(v, q)$. We keep the parameter condition as in the previous lecture:

- $q$ is a primitive $e^{th}$-root of unity, which is encoded in the type of $\mathfrak{g}$.
- the parameters $v_i$ are $v_i = q^{\gamma_i}$, which are encoded in the highest weight $\Lambda$.

We identify $K_0(\mathcal{H}_*(v, q)\text{-mod})$ with $\mathcal{F}$ via

$$S^\lambda \longleftrightarrow \lambda.$$

We also make $K_0(\mathcal{H}_*(v, q)\text{-proj})$ into a $\mathfrak{g}$-module by using refined version of induction/restriction functors.

Then the theorem is as follows.
Theorem 2 (A). The map

\[ t_{dec}^* : K_0(\mathcal{H}_*(v, q)\text{-proj}) \to K_0(\mathcal{H}_*(v, q)\text{-mod}) \]

is naturally a \( g(A_{e-1}) \)-module homomorphism and we have.

1. \( \text{Im} \ t_{dec}^* = V(\Lambda) \subset \mathcal{F} \).
2. If \( \text{char } F = 0 \) or \( G(\mu) \) is of the form
   \( (\text{divided power product}) \times \emptyset \), then
   \[ t_{dec}^*([P^\mu]) = \sum_{\lambda} d_{\lambda \mu} \lambda = G(\mu) \]
   \( (\text{for } \mu \in B(\Lambda) \subset \mathcal{P}_{\text{tensor}}^m) \).

where \( P^\mu \) is the projective cover of \( D^\mu \).

Recall that using algebraic \( K \)-theory, we have an isomorphism of \( \mathcal{R}(G \times \mathbb{C}^\times) \)-algebras

\[ K_0^{G \times \mathbb{C}^\times}(Z) \cong \widehat{\mathcal{H}}_W(q) \]

where \( Z \) is the Steinberg variety, \( \widehat{\mathcal{H}}_W(q) \) is associated with \( (W, q, \mathbb{Z}[q, q^{-1}]) \), by the work of Ginzburg and Lusztig.
Then, after going down to $K^A(Z)$, where $A$ is the minimal closed algebraic subgroup that contains $(s, q) \in T \times \mathbb{C}^\times$, using Thomason's concentration theorem, we can specialize the center $R(G \times \mathbb{C}^\times)$ to $\mathbb{C}$, by evaluation at $(s, q)$.

Modifying the linear isomorphism

$$K(Z^{(s,q)})_{\mathbb{C}} \cong CH(Z^{(s,q)})_{\mathbb{C}} \cong H^{BM}_*(Z^{(s,q)}, \mathbb{C})$$

we get the algebra isomorphism

$$\mathbb{C} \otimes_{R(G \times \mathbb{C}^\times)} \mathcal{H}_W(q) \cong H^{BM}_*(Z^{(s,q)}, \mathbb{C}).$$

Going further to sheaf theoretic analysis, this geometric theory allows us to handle the simple modules on the Grothendieck group level.

However, as we want to know is the relationship between the Specht modules and the simple modules, not the relationship between the standard modules and the simple modules, we need some arguments.
Also, this theory depends on the idea that

“Reduce the center first”.

But, if we do so, we cannot see the existence of the crystal structure in the theory. Thus, $H_n(v, q)$ is a better quotient of the affine Hecke algebra of type $A$, from our point of view.

An application and another approach to it

Combined with Leclerc’s work that the decomposition numbers of the $q$-Schur algebra appear in those of $H_{S_n}(q)$, a theorem of Dipper and James says that we can compute the decomposition numbers of $GL_n(\mathbb{F}_q)$ in non-defining characteristic cases by using our result.

By a theorem of Varagnolo and Vasserot, we can compute them by another method of using the Leclerc-Thibon canonical basis for the full deformed Fock space.

In both, we assume James’ conjecture on the lower bound.
§3 Representation type

Another application of the previous theorem is about the representation type of \( \mathcal{H}_W(q) \).

**Definition 3.** Let \( A \) be a finite dimensional \( F \)-algebra. We say that \( A \) is of

- **finite type** if the number of indecomposable \( A \)-modules is finite.
- **tame type** if it is not of finite type and, for each \( d \in \mathbb{N} \), there exist
  \((A, F[X])\)-bimodules \( M_1, \ldots, M_{n_d} \)
  which is free of finite rank as an \( F[X] \)-module, such that all but a finite number of indecomposable \( A \)-modules of dimension \( d \) are of the form
  \[
  M_i \otimes_{F[X]} F[X]/(X - \lambda).
  \]
wild type if there exists an \((A, F\langle X, Y \rangle)\)-module \(M\) which is free of finite rank as an \(F\langle X, Y \rangle\)-module such that the functor \(M \otimes - : F\langle X, Y \rangle\text{-mod} \to A\text{-mod}\) respects indecomposability and isomorphism classes.

By the famous theorem of Drozd, Crawley-Boevey, \(A\) is finite, tame or wild, and they are mutually exclusive.

This is the representation type of \(A\).

Let \(A\) be a finite dimensional \(F\)-algebra as above, \(\{P_1, \ldots, P_n\}\) the set of PIM’s. Then \(A\) is Morita-equivalent to \(B = \text{End}_A(\bigoplus P_i)^{\text{op}}\), and the latter has the form

\[ B = FQ/I \]

by a result of Gabriel.
$Q$ is a quiver (digraph), and $I$ is an admissible ideal of the path algebra $FQ$. That is, $I$ is contained in the ideal consisting of the paths of length $\geq 2$ and contains all the paths of length $\geq N$ for some $N$.

The path algebra with relations $FQ/I$ has been studied intensively. In particular, there are many results to tell the representation type of $FQ/I$.

In our application to Hecke algebras,

Fock space theory $+$ Specht module theory
\[\Downarrow\]
we can control the structure of $B = FQ/I$

to some degree in crucial cases,

which suffices to determine the representation type of $\mathcal{H}_W(q)$ for $W$ whose components are free of exceptional type.
Theorem 4 (A). Let $W$ be a finite Weyl group without exceptional components, and assume that $F$ is an algebraically closed field. Then $\mathcal{H}_W(q)$ is

(1) semisimple if and only if $P_W(q) \neq 0$.

(2) finite and not semisimple if and only if $x - q \mid P_W(x)$.

(3) tame if and only if $q = -1 \neq 1$ and $(x + 1)^2 \mid P_W(x)$.

(4) wild if otherwise.

The first is wellknown. The second has settled Uno’s conjecture for classical types.

Note that only a finite number of cases remain to be settled. Namely the cases in exceptional types. H.Miyachi has some computation for these cases. We expect:

If $\text{char } F$ is almost good then the theorem still holds.
For $\mathcal{H}_{S_n}(q)$, we have the block version of the result, due to Erdmann and Nakano.

Recall that Dipper and James determined how $\mathcal{H}_{S_n}(q)$ is decomposed into block algebras. The labelling is by $e$-cores.

**Definition 5.** Let $\kappa$ be a partition. If we cannot remove a rimhook of length $e$ from $\kappa$, we say that $\kappa$ is an $e$-core.

The block algebras of $\mathcal{H}_{S_n}(q)$ are labelled by $e$-cores $\kappa$ with $n - |\kappa| \geq 0$ divisible by $e$.

The number $w = \frac{n - |\kappa|}{e}$ is called the **weight** of the block algebra.
Theorem 6 (Erdmann and Nakano). Let $B_\kappa$ be the block algebra of $\mathcal{H}_{S_n}(q)$ labelled by an $e$-core $\kappa$. Write $w_\kappa$ for its weight. Then $B_\kappa$ is

1. semisimple if and only if $w_\kappa = 0$.
2. finite if and only if $w_\kappa \leq 1$.
3. tame if and only if $q = -1 \neq 1$ and $w_\kappa = 2$.
4. wild if otherwise.

It is important to obtain the similar results for other types.

Hecke algebras share many properties with group algebras. However, group algebras are Hopf algebras and Hecke algebras do not have natural Hopf algebra structure.

Because of this, the analogy works to guess what kind of theorems must hold for Hecke algebras, but the analogy does not work to find their proofs. Uno’s conjecture is such an example. The analogy is called the q-analogue philosophy.
Tree classes

I have explained the q-analogue philosophy: it produces many good problems by guessing the counterparts of the group algebra properties. I would like to explain an open problem coming from this philosophy.

If an algebra is wild then we cannot expect the classification of indecomposable modules. However, there is still a hope to consider some kind of the classification problem, as the set of indecomposable modules is not a mere set, but has a graph structure.

Recall that $\mathcal{H}_W(q)$ is a symmetric algebra. Thus, by deleting the nodes of projectives we get the stable Auslander-Reiten quiver, which we now define.
Definition 7. Let $A$ be a finite dimensional $F$-algebra, $M$ and $N$ indecomposable, and

$$f : M \to N$$

an $A$-module homomorphism.

$f$ is called **irreducible** if

– $f$ is neither a split mono nor a split epi.
– if $f$ is factored as $f : M \xrightarrow{g} L \xrightarrow{h} N$
  then either $g$ is a split mono or $h$ is a split epi.

Definition 8. Let $A$ be a self-injective $F$-algebra. The **stable Auslander-Reiten quiver** is

the $\mathbb{Z}^2_{\geq 0}$-colored oriented graph $\Gamma_s(A)$ whose
– vertices are non-projective indecomposable $A$-modules,
– arrows are irreducible homomorphisms.
The color \((a, b)\) on an edge is defined by using the Auslander-Reiten sequences.

The stable AR-quiver has the structure of a translation quiver.

Let \(Q\) be a connected component as a translation quiver. Then the Riedmann structure theorem asserts that there is a unique tree \(T\) and an admissible group action

\[ G \times \mathbb{Z}T \to \mathbb{Z}T \]

such that \(Q \simeq \mathbb{Z}T/G\) as a translation quiver.

\(T\) is called the \textit{tree class} of \(Q\).

Erdmann showed that if a block algebra \(B\) of a group algebra \(FG\) is of wild type then any connected component \(Q\) of \(\Gamma_s(B)\) has the tree class \(A_\infty/2\).

\textbf{Problem} Prove the same result for \(\mathcal{H}_W(q)\).